A principle of relativity for quantum theory

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Abstract

In non relativistic physics it is assumed that both chronological ordering and causal ordering of events (telling whether there exists a causal relationship between two events or not) are absolute, observer independent properties. In relativistic physics on the other hand chronological ordering depends on the observer who assigns space-time coordinates to physical events and only causal ordering is regarded as an observer independent property. In this paper it is shown that quantum theory can be considered as a physical theory in which causal (as well as chronological) ordering of probabilistic events happening in experiments may be regarded as an observer dependent property.

1 Introduction

The most notable attempts in formulating a theory that unifies quantum theory and general relativity are String Theory and Loop Quantum Gravity [1, 2]. The lack of experiments that could verify or falsify any of the predictions of the two theories leaves physicists with the consciousness that something is missing in our current understanding of nature at the fundamental level. Despite the formulation of both theories mentioned above depart from very reasonable starting points, they remain naive about giving a foundational principle to explain the mathematical formalism of quantum theory.

This means that they retain superposition principle, non-locality and all the counterintuitive features manifested by quantum theory as natural facts and do not try to give a motivation for them. This attitude is perhaps justified by the fact that quantum theory is extremely successful in making predictions. Until now, no experimental situation has been found in which the predictions of quantum theory are not satisfied. Such an extraordinary predicting power has led many physicists to think that it is not necessary to have a physical intuition of what is going on at atomic and subatomic scales, it is sufficient to have a model that can predict whatever can occur in an experiment. This pragmatic attitude would be the right one if theoretical physics accomodated all phenomena experienced in nature in a unique and coherent model. Despite the many successes of the Standard Model and the potentiality of string theory and loop quantum gravity, there is large consensus among physicists that we are far away to have such a unified picture. This has recently led physicists to turn the attention back to the problem of foundations quantum theory with a new slant given by the emergence of quantum information [3, 4, 5, 6, 7, 8, 9]. In this paper it is analyzed the mathematical structure of quantum theory (as used in the filed of quantum information) from a novel point of view enlighting the interplay between quantum features and causal structure of space-time events. In quantum theory events correspond to probabilistic outcomes and the only predictable and verifiable statements regard correlations between outcomes happening on different devices located in distinct regions of space. Since these outcomes are also thought to be events happening in space-time, it is always assumed an absolute causal ordering for them. A set of physical events $\mathcal{E}$, like those that can happen in a quantum experiment, possesses a causal ordering if, for these events, it is defined a *causal structure*. This means that for any pair of events, $\chi_a, \chi_b \in \mathcal{E}$, one of the following must hold:

- χ_a causes χ_b
- χ_b causes χ_a
- χ_a does not cause χ_b and χ_b does not cause χ_a (they are space-like events)

For example, χ_a could be a preparation contained in a preparations ensemble for a quantum system of a certain type while χ_b could be an outcome of a measurement caused by that preparation. In this case χ_a causes χ_b. χ_a and

χ_b could also be two outcomes obtained respectively in two measurements performed in parallel on a bipartite state of a composite system. In this case χ_a and χ_b are indeed two space-like separated events. The main result of this paper is that, in quantum theory, any experimental situation of the former type mentioned above can be considered as equivalent to a situation of the latter type. This equivalence is such that the two experiments can be interpreted as the same experiment viewed by two different observers that makes two different assumptions regarding the causal ordering of events happening in the experiment. To prove this it is shown in section 3 that, in a generic quantum experiment involving two sets of random outcomes happening on distinct devices, the mathematical expression of the joint probability of any two outcomes calculated by one observer, can be mapped, by means of a simple transformation rule, into the expression for the joint probability of the same two outcomes calculated by another observer that assumes a different causal ordering of events with respect to the first. After having generalized this concept to experiments involving more sets of random outcomes we are led to introduce a new physical principle, the "Principle of Relativity of Causal Structure", and to put it as a foundational principle for quantum theory. From this principle we understand that a possible way to move towards a theory of quantum gravity is to retain causal structure of physical events as an observer dependent property. Here we take a first step in this direction comparing the idea that causal structure is an observer dependent property with the role causal structure plays in general relativity (see section 4). It is argued that the situation in general relativity theory is somewhat opposite to the one outlined in quantum theory. If, in quantum theory, causal ordering of probabilistic events can be regarded as an observer dependent property, this clearly cannot hold in general relativity. In general relativity, the causal ordering of two events is represented by the value of the metric function evaluated at the two space-time points representing those events. Einstein's equations relate the metric function to the stress-energy tensor representing energy density in the portion of universe including the two events. This implies that, in general relativity, whether it exists a causal influence between two events or not, ultimately depends on energy density that is an objective, physically measureable quantity and hence cannot be regarded as an observer dependent property. Elevating the principle of relativity of causal structure to universal principle finally leads us to consider dark energy not as a conceptual problem but as an essential ingredient of our current understanding of the universe (see section 5).

This research is important for two reasons. The first is that it gives a new foundational principle to motivate the mathematical structure of quantum theory. The second is that, in doing this, it is possible to argue that one of the most puzzling features of modern theoretical physics, dark energy, could be explained elevating the above foundational principle for quantum theory to a universal principle. Clearly this would imply that Einstein's theory of general relativity should be definitely abandoned and should be elaborated a deeper theory of the cosmos to explain observational data.

2 Space-time and causal structure

A space-time is, roughly speaking, a mathematical representation of physical events. For any set of physical events $\mathcal{E}$, given two events $p, q \in \mathcal{E}$ one of the three mutually exclusive alternatives must hold:

- p is the cause of q
- q is the cause of p
- p is not the cause of q and viceversa.

Specifying one of the three alternatives for every pair of events leads to define the *causal structure* of the set $\mathcal{E}$. The first of the above alternatives means that q is in the future of p, the second means that p is in the future of q while the third means that it is impossible for a physical system to be present in correspondence with both events p and q (i.e. p and q are causally independent).

In non relativistic (or newtonian) space-time, given an event p for all other events q it must hold one of the following alternatives: (i) q is in the future of p; (ii) q in the past of p (iii) q happens at the same time of (is simultaneous with) p. Regarding this latter case, the events simultaneous with p constitute points of a three dimensional euclidean space. This distinction comes from the fact that, in non relativistic space-time, the chronological ordering of events is the same as their causal ordering. If p and q are one the cause of the other then necessarily one must happen before the other while if p and q are causally independent then they must necessarily happen at the same time.

In relativistic space-time the latter fact above mentioned does not hold anymore. In particular, two causally independent events can be simultaneous

for some observers and have a different chronological ordering for another observer. From this fact the set of events $q \in \mathcal{E}$ that constitutes the past and future of p are represented respectively as points of a four dimensional cone while the set of events that are not in past nor in the future of p are represented by points outside those two cones embedded in euclidean four dimensional space.

Both in non relativistic and relativistic physics, two different observers can in principle assign different coordinates to a physical event p because they move relatively to one another. In newtonian space-time if observer O labels p with coordinates (t, x, y, z) and O' moves with velocity v in the x direction passing O at $t = x = y = 0$ then the coordinate labels assigned to p by O' are $t' = t, x' = x - vt, y' = y, z' = z$. In special relativity, i.e. if v is sufficiently close to the speed of light c, those relations become $t' = (t - vx/c^2)/(1 - v^2/c^2)^{1/2}), x' = (x - vt)/(1 - v^2/c^2)^{1/2}), y' = y, z' = z$. Since two different observers looking at the same physical process must describe the same physics independently of their state of motion relative to one another, it is clear that the above transformations of coordinates leave unaffected any significant physical property. This implies that coordinate labels do not have any intrinsic physical significance since they only depend on which observer labels physical events.

The causal structure of any set of events $\mathcal{E}$ is incorporated in any space-time that can be used to represent those events. Moreover, it constitutes an absolute, observer independent property, contrary to the space-time coordinates assigned to them. For this reason, in both newtonian and relativistic space-time there exist specific functions of the coordinates of any two points p and q, that remain unchanged in changing point of view from one observer to another. In newtonian physics this function is the time interval $\Delta_t = t_p - t_q$. In special relativity this function is $M = -(\Delta t)^2 + 1/c^2[(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2)]$. In general relativity this function is represented by the metric tensor associated to a manifold representing a solution of Einstein's equations. The value of these functions evaluated at every pair of points (p, q) encodes the causal structure of events.

We can thus say that both newtonian and relativistic space-time are different mathematical ways to model a set of events with an absolutely (i.e. independently of observers) defined causal structure.

Outcomes happening on devices in quantum experiments are supposed to be events in space-time. From this fact they possess a definite, observer independent causal structure. In the next section we are going to show that,

although an absolute causal structure of events is a background assumption in the usual formulation of quantum theory, the quantum formalism permits to compute correlations for events happening in experiments in such a way that their causal structure can be regarded as an observer dependent property.

3 Causal structure in quantum theory

In what follows we are going to show that causal structure in quantum theory may be regarded as an observer dependent property rather than fixed in an absolute way. We will do this first considering a specific situation from quantum optics and then generalizing it. Based on this result, we will then state two principles, one of which is called relativity of causal structure, that can be posed as foundational principles for quantum theory.

Consider the quantum experiment involving polarized photons shown in figure 1.

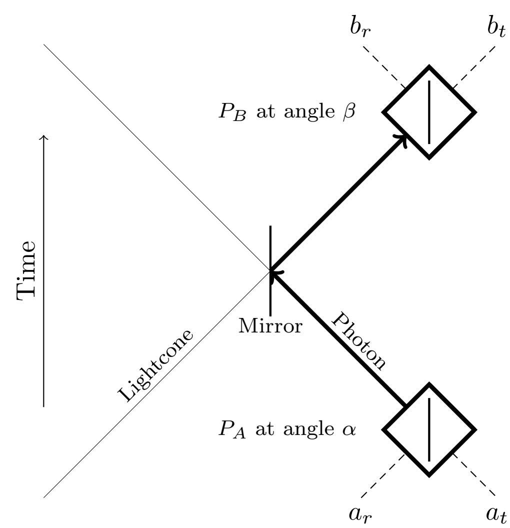

Figure 1

We have two polarizers P_A and P_B, the former aligned at an angle α and the latter aligned at an angle β. A photon passes first through P_A is reflected by a mirror and then passes through P_B. For the experiment to take place

the photon must either be transmitted or be reflected by polarizer P_A. Hence associated to P_A we have two possible mutually exclusive outcomes that we indicate $\{a_r, a_t\}$. After the mirror reflection the photon enters P_B and then is absorbed by some photon counter. In order to be counted the photon must either be transmitted or be reflected by P_B. Hence also associated to P_B we have two mutually exclusive outcomes that we call $\{b_r, b_t\}$. The information contained in the experiment is represented by the joint probability distribution $p(a_i, b_j)$ with $(a_i, b_j) \in \{a_r, a_t\} \times \{b_r, b_t\}$. The arrows linking the various devices represent the path followed by the photon. In particular the arrow pointing out of P_A means that the photon is an output system for polarizer P_A. The arrow pointing inside P_B means that the photon is an input system for P_B. The lightcone and the arrow of time are drawn to remark that two events associated to any pair of outcomes (a_i, b_j) are one the cause of the other. Indeed there is a physical system, i.e. the photon, that carries the information regarding the probability distribution $\{p(a_i)\}_{a_i \in \{a_r, a_t\}}$ from P_A to P_B. This means that if the probability distribution $\{p(a_i)\}_{a_i \in \{a_r, a_t\}}$ changes and becomes $\{q(a_i)\}_{a_i \in \{a_r, a_t\}}$ then also the probability distribution $\{p'(b_j)\}_{b_j \in \{b_r, b_t\}}$ changes. The above discussion implies that any pair of outcomes (a_i, b_j) is such that a_i causes b_j and the correlations between the sets of random outcomes $\{a_i\}$ and $\{b_j\}$ are due to a causal influence.

Consider now the experiment shown in figure 2.

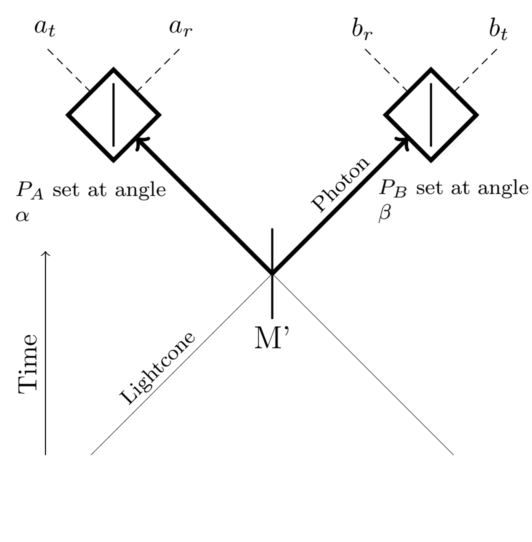

Figure 2

We have the same polarizers P_A and P_B involved in the previous experiment and for simplicity we assumed they are aligned in the same direction as before. Two photons in an entangled state of zero total angular momentum start from a source of entangled photons, M', and reach independently P_A and P_B respectively. After they have passed the polarizers they are absorbed by two photon counters placed after P_A and P_B respectively. For the experiment to take place, both the photons must be either transmitted or reflected by the respective polarizers before being detected. Hence also in this case, associated to both P_A and P_B, there are two sets mutually exclusive outcomes $\{a_r, a_t\}$ and $\{b_r, b_t\}$ and these represent the same outcomes as in the previous experiment. The joint probability distribution $p(a_i, b_j)$ with $(a_i, b_j) \in \{a_r, a_t\} \times \{b_r, b_t\}$ contains the information about the experiment. In figure 2 there are two arrows pointing inside polarizers P_A and P_B respectively. Also in this case are drawn the lightcone and the arrow of time to help visualizing that any pair of outcomes $(a_i, b_j) \in \{a_r, a_t\} \times \{b_r, b_t\}$ represents two space-like events.

The two experiments described above seem very different. The latter involves, for each repetition of the experiment, a pair of entangled photons while the former involves a single photon. This difference in their physical description is due to the fact that in each run of the experiment, it is assumed in one case that the pair of outcomes (a_i, b_j) are one the cause of the other (the casual relationship being represented by a photon travelling from P_A to P_B) and in the other case that they are two space-like events (since they are due to two causally independent systems). We can thus say that the main difference in the two above experiments relies on how, each run of the experiment, the outcomes $(a_i, b_j) \in \{a_i\}_{i=r,t} \times \{b_j\}_{j=r,t}$ are embedded in space-time. The setup in figure 1 involves three devices, the two polarizers P_A and P_B and a mirror M. The experiment in figure 2 also involves three devices, two of them are the same polarizers as before while the third device, M' is a source of entangled photons. For the experiment in figure 1 the photon is an output system for P_A, it is an input and an output for M while it is an input system for P_B. For the experiment in figure 2 the photons involved may be regarded as two outputs for M' and as two input systems for P_A and P_B respectively. Hence the difference between the two experiments is that a photon is seen as an output system for P_A (and in consequence as an input for M) in the experiment of figure 1 while it is seen as an input system for P_A (and in consequence as an output for M') in the experiment of figure 2. From the above discussion we can say that the existence of a

causal relationship between the region where lies P_A (where happen outcomes $\{a_r, a_t\}$) and the region where lies P_B (where happen $\{b_r, b_t\}$) is equivalent to assign a specific input/output structure for the devices involved in the experiment. We can thus say that the input/output structure of the devices involved in the experiment is equivalent to the causal structure assigned to the outcomes associated to those devices.

In both the situations described above it is assumed a definite causal structure between the region of space where lies P_A and that where lies P_B. This means that it is assumed in an absolute way either that between region P_A and region P_B there exists a causal relationship or that regions P_A and P_B are space-like separated. On the other hand, every experiment in quantum theory is intrinsically probabilistic and whatever an observer might experience reduces to correlations between outcomes happening on two devices in distinct regions. This observation suggests that a definite causal structure between region P_A and region P_B could not be significant in predicting joint probabilities for events happening in these two regions. Since correlations between events is the only observable and physically predictable property in quantum theory, it could be the case that the two experiments described in figure 1 and 2 are simply a different way to describe the same experiment. Indeed they both define a joint probability distribution between the values of the same pair of observables (polarizations along α and β), they refer to the same type of system (the photon) and differ only because in the former it is assumed a causal relationship between regions P_A and P_B while in the latter it is assumed that regions P_A and P_B are space-like separated. In what follows we will show that the mathematical formalism of quantum theory is consistent with the above suggestion.

Suppose that an experimenter sets up one of the two experiments illustrated above, say the one in figure 1 for definiteness. Two observers look at this experiment without knowing the nature of device M and the actual input/output structure between the devices. The observers experience the correlations between the set of outcomes $\{a_r, a_t\}$ associated to P_A and the set $\{b_r, b_t\}$ associated to P_B. To one observer it is said that M is a mirror and that the setup is actually the one in figure 1. To the other observer it is indeed said that M constitutes a source of maximally entangled photons and that the setup corresponds to the one in figure 2. We will call the former observer O_1 and the latter observer O_2. Comparing figure 1 and 2 we can readily understand that O_1 assumes that photons constitute outputs for P_A and inputs for P_B while O_2 assumes that photons constitute

inputs for both P_A and P_B. These two assumptions cannot be verified (or falsified) by the two observers experiencing correlations between $\{a_r, a_t\}$ and $\{b_r, b_t\}$. Hence they can calculate the joint probability distribution $\{p(a_i, b_j)\}$ with $(a_i, b_j) \in \{a_r, a_t\} \times \{b_r, b_t\}$ on the base of the information they respectively have regarding causal structure. We will now show that for all $(a_i, b_j) \in \{a_r, a_t\} \times \{b_r, b_t\}$, the probability calculations of observers O_1 and O_2, although apparently different, reduce to the same calculation and give rise to the same probability value. According to this we may conclude that the two experiments in figures 1,2 are the same experiment seen by two different observers who assume a different causal structure between the regions where are situated polarizer P_A and P_B.

O_1 assumes that the polarizer P_A prepares an ensemble represented by $p|a_r\rangle\langle a_r| + (1-p)|a_t\rangle\langle a_t|$. By now, let us assume $p = 1/2$ for simplicity. The probability of seeing outcome b_t in correspondence of P_B given that it is prepared a photon in state a_r is $p(b_t|a_r) = |\langle b_s|a_r\rangle|^2$ thus the joint probability is:

$$p(a_r, b_t) = 1/2\langle b_t|a_r\rangle^2 \tag{1}$$

O_2 indeed assumes that M is a source of entangled photons in state $\psi = 1/\sqrt{2}|a_r a_r\rangle + |a_t a_t\rangle$. The joint probability of seeing outcomes a_r and b_t calculated by O_2 is:

$$p(a_r, b_t) = |\langle a_r| \otimes \langle b_t| 1/\sqrt{2}(|a_r a_r\rangle + |a_t a_t\rangle)|^2 \tag{2}$$

But the above equation actually reduces to (1). Expliciting (2) we have:

$$p(a_r, b_s) = 1/2(\langle a_r|a_r\rangle^2\langle b_t|a_r\rangle^2 + \langle a_r|a_t\rangle^2\langle b_t|a_t\rangle^2 + 2\langle a_r|a_r\rangle\langle b_t|a_r\rangle\langle a_r|a_t\rangle\langle b_t|a_t\rangle) \tag{3}$$

and all terms in (3) are zero except the first thus we can write:

$$p(a_r, b_t) = 1/2\langle b_t|a_r\rangle^2 \tag{4}$$

Clearly the above reasoning is true for every pair $(a_i, b_j) \in \{a_r, a_t\} \times \{b_r, b_t\}$. Moreover it is simple to convince ourselves that nothing would change if we assumed that the set up prepared by the experimenter at which O_1 and O_2 both look was that in figure 2 in place of the one in figure 1. This simple example shows that the assumptions of O_1 and O_2 regarding causal structure of regions P_A and P_B are equivalent for the purpose of calculating joint probabilities. Whatever an observer of anyone of the above experiments can experience are correlations between outcomes in region P_A and outcomes

in region P_B, and whatever he can predict are joint probabilities for the outcomes in those regions. Hence, the fact that between those two regions there exists a causal relationship or not is a property that depends on the assumption of an observer and cannot be fixed absolutely for all observers in any way.

Note that the equivalence stated above derives from he fact that (2) is an alternative way of writing (1). If it were not so then causal structure could not be an observer dependent property. Indeed the correlations between region P_A and region P_B depend on the probability distribution $\{p(a_i, b_j)\}$ and if the probability distribution calculated by observer O_2 was different from that calculated by observer O_1 then one of the observers, O_2, would predict wrong probabilities and would become aware, after camparing his calculations with those of O_1, that correlations are effectively due to a causal relationship between P_A and P_B. This implies that the equivalence of the two above situations is a consequence of how in quantum theory are performed probability calculations for the experiments illustrated in figure 1 and 2.

The two situations considered above are far from being the most general experiments correlating random outcomes in two regions of space. The equivalence of (1) and (2) could infact be a numerical coincidence. In the remaining part of this section we will prove that the above property is a general feature of quantum theory. We will consider a generic quantum experiment in which two devices D_A and D_B display two sets of random outcomes $\{a_i\}_{i\in X}$ and $\{b_j\}_{j\in Y}$ respectively with X and Y two sets of outcomes. The information on such correlations is contained in the joint probability distribution $\{p(a_i, b_j)\}_{(i,j)\in X\times Y}$. As in the previous example, we suppose that two observers O_1 and O_2 are looking at the experiment; O_1 assumes that correlations between D_A and D_B are due to a system causally correlating the outcomes in $\{a_i\}_{i\in X}$ to those in $\{b_j\}_{j\in Y}$ while O_2 assumes that D_A and D_B lie in space-like separated regions.

Observer O_1

O_1 assumes that correlations are due to a causal relationship. In this case a system $\mathscr{S}$ carries the information of the probability distribution of one of the sets of outcomes, say $\{a_i\}_{i\in X}$ with probability distribution $\{p_i\}_{i\in X}$, from device D_A to device D_B. The experiment seen by O_1 is represented in figure 3.

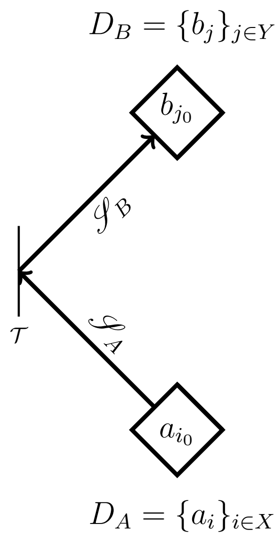

Figure 3

System $\mathscr{S}_A$ is the output system for D_A while $\mathscr{S}_B$ is the input system for D_B. We call them with different names to mantain full generality. Of course they could be the same type of system. An outcome $a_{i_0} \in \{a_i\}_{i\in X}$ is a preparation belonging to the preparations ensemble $\{a_i\}_{i\in X}$ with associated probability ditribution $\{p_i\}_{i\in X}$. The ensemble is represented by a density matrix ρ and a POVM $\{\mathbf{a_i}\}_{\mathbf{i\in X}}$ as follows:

$$\rho = \sum_{i\in A} \mathrm{Tr}[\mathbf{a_i}\rho]\frac{\sqrt{\rho}\,\mathbf{a_i}\sqrt{\rho}}{\mathrm{Tr}[\mathbf{a_i}\rho]} \tag{5}$$

To achieve as much generality as we can, we will not make any restriction on ρ a part from assuming that it does not represent a pure state since otherwise the outcomes on device D_A would not be random anymore contrary to our initial assumptions. The ensemble ρ causes probabilistically an outcome $b_{j_0} \in \{b_j\}_{j\in Y}$ on device D_B. In the most general case, this is represented by an element of a POVM $\{\mathbf{b_j}\}_{\mathbf{j\in Y}}$ for system $\mathscr{S}_B$. The ensemble represented by ρ before causing outcome b_{j_0} will eventually undergo an evolution that is generically represented by a Completely Positive Trace Preserving (CPTP) map $\mathscr{T}$. Its Kraus decomposition is $\sum_m K^m \otimes K^{m\dagger}$ with $K^m = \sum_{ef} K^m_{ef} |e\rangle_{BA}\langle f|$ Kraus operator [10] ($\{|e\rangle\}_{e=1}^{d_B}, \{|f\rangle\}_{f=1}^{d_A}$ are orthonormal basis for hilbert space of $\mathscr{S}_B$ and $\mathscr{S}_A$ respectively). We now explicit the

evolution of ensemble ρ by means of transformation $\mathscr{T}$. The density matrix obtained after the evolution is:

$$\mathscr{T}(\rho) = \sum_{m,ef,cd} K^m_{ef} K^{m*}_{cd} |e\rangle_{BA}\langle f|\rho|c\rangle_{AB}\langle d| \tag{6}$$

Using the fact that $\sum_m K^m \otimes K^{m\dagger}$ can be written as:

$$\sum_{m,ef,cd} K^m_{ef} K^{m*}_{cd} |c\rangle_{AA}\langle f| \otimes |e\rangle_{BB}\langle d| \tag{7}$$

and the polar decomposition of ρ we have:

$$\mathscr{T}(\rho) = \mathrm{Tr}_A[\sum_{m,ef,cd} K^m_{ef} K^{m*}_{cd} \sqrt{\rho}|c\rangle_{AA}\langle f|\sqrt{\rho} \otimes |e\rangle_{BB}\langle d|] \tag{8}$$

Note that, for the polar decomposition of ρ to be uniquely defined, one must assume ρ to be full rank in the Hilbert space corresponding to $\mathscr{S}_A$. The density matrix obtained after the evolution can thus be written as $\mathscr{T}(\rho) = \mathrm{Tr}_A[\mathscr{T}_\rho]$ where we define:

$$\mathscr{T}_\rho := \sqrt{\rho} \otimes I_B[\sum_m (K^m \otimes K^{m\dagger})]\sqrt{\rho} \otimes I_B \tag{9}$$

where I_B is the identity matrix on system $\mathscr{S}_B$. From (9) we see that the evolution of ensemble ρ can be represented as an operator acting on Hilbert spaces of systems $\mathscr{S}_A$ and $\mathscr{S}_B$. The probability calcualted by observer 1 is then:

$$p_1(a_{i_0}, b_{j_0}) = \mathrm{Tr}_B[\mathbf{b_{j_0}} \mathrm{Tr}_{\mathbf{A}}[\mathscr{T}_\rho \mathbf{a_{i_0}}]] \tag{10}$$

Observer O_2

O_2 indeed assumes that correlations are not due to a causal relationship. This means that the two sets of outcomes constitute two measurements performed in parallel on two copies of system $\mathscr{S}$. In figure 4 it is represented the same experiment of figure 3 as seen by observer O_2 assuming that the regions in which are situated D_A and D_B are space-like separated.

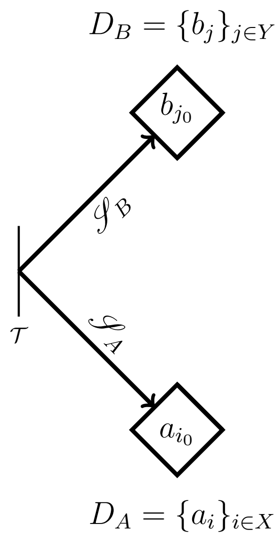

Figure 3

Systems $\mathscr{S}_A, \mathscr{S}_B$ constitute two causally independent inputs for devices D_A and D_B. The two systems are both outputs of a common source denoted as τ in the above figure. This can be represented by a bipartite state τ_{AB} that permits the observer to calculate the joint probability $p(a_{i_0}, b_{j_0})$ for all pairs of outcomes as follows:

$$p_2(a_{i_0}, b_{j_0}) = \mathrm{Tr}_{AB}[\mathbf{a_{i_0}}' \otimes \mathbf{b_{j_0}}' \tau_{\mathbf{AB}}] \tag{11}$$

where $\mathbf{a_{i_0}}'$ and $\mathbf{b_{j_0}}'$ are elements of the POVMs $\{\mathbf{a_i}'\}_{\mathbf{i}\in\mathbf{X}}$, $\{\mathbf{b_j}'\}_{\mathbf{j}\in\mathbf{Y}}$ corresponding respectively to outcomes a_{i_0}, b_{j_0}.

Assumptions of observers O_1 and O_2 are equivalent

We are now going to prove the following statement: Given the mathematical objects used to calculate joint probabilities of the outcomes by O_1, there exists a unique choice of mathematical objects that permits O_2 to calculate the same joint probability distribution of outcomes. Before proving the above statement we recall the discussion regarding the equivalence between input/output structure and causal structure in quantum experiments. The only difference between the experiment seen by O_1 and the experiment seen by O_2 is that system $\mathscr{S}_A$ is assumed as an output for D_A by O_1 while is assumed as input for D_A by O_2. This becomes apparent comparing figure 3

with figure 4. Based on this observation, we now give the rule that permits to prove the statement done at the beginning of this paragraph.

> **Transformation Rule:** If a system $\mathscr{S}$, with hilbert space $\mathcal{H}_{\mathscr{S}}$ is an input (output) for O_1 and an output (input) for O_2, then the operators involving $\mathcal{H}_{\mathscr{S}}$ used by O_1 are the transposed on $\mathcal{H}_{\mathscr{S}}$ of those used by O_2.

From the above rule, if $\mathbf{a_{i_0}}$ represents an element of the preparation ensemble ρ of O_1, $\mathbf{a_{i_0}^T}$ represents the corresponding measurement outcome for O_2. For the same reason, the bipartite state τ_{AB} has the following expression:

$$\tau_{AB} = \mathscr{T}_{\rho}^{T_A} = \sqrt{\rho}^T \otimes I_B[\sum_m (K^m \otimes K^{m\dagger})^{T_A}]\sqrt{\rho}^T \otimes I_2 \tag{12}$$

Where ^{T_A} denotes partial transposition on hilbert space $\mathcal{H}_{\mathscr{S}_A}$. First we have to prove that (12) is a normalized bipartite state. This can be seen defining the normalized bipartite state on two copies of $\mathscr{S}_A$, $|\Phi\rangle_{AA'}$:

$$|\Phi\rangle_{AA'} = \sqrt{\rho}^T \otimes I_{A'} \sum_j |j\rangle_A \otimes |j\rangle_{A'} \tag{13}$$

where $\{|j\rangle\}_{j=1}^{d_A}$ is an orthonormal basis for space $\mathcal{H}_{\mathscr{S}_A}$. Exploiting (13) we can write:

$$\mathscr{I} \otimes \mathscr{T}(|\Phi\rangle\langle\Phi|) = \tau_{AB} \tag{14}$$

where $\mathscr{I}$ is the identity map on $\mathcal{H}_{\mathscr{S}_A}$ and $\mathscr{T}$ represents the evolution defined above. From (14) we can see that τ_{AB} is a normalized bipartite state since $\mathscr{T}$ is a TPCP map acting on system $\mathscr{S}_A$ and $|\Phi\rangle\langle\Phi|$ is a normalized bipartite state. The probability $p_1(a_{i_0}, b_{j_0})$ expressed in (10) calculated by O_1 is then equal to the probability $p_2(a_{i_0}, b_{j_0})$ calculated by O_2, namely:

$$p_2(a_{i_0}, b_{j_0}) = \mathrm{Tr}_{AB}[\mathbf{a_{i_0}}^T \otimes \mathbf{b_{j_0}} \mathscr{T}_{\rho}^{T_A}] = p_1(a_{i_0}, b_{j_0}) \tag{15}$$

This expression represents the probability for a given pair of outcomes $(a_{i_0}, b_{j_0}) \in \{a_i, b_j\}_{(i,j)\in X\times Y}$ to jointly happen. This proves the statement done at the beginning of this paragraph.

In conclusion, every experiment in quantum theory is intrinsically probabilistic and whenever it correlates two sets of random outcomes displayed by two devices in two distinct regions of space, an observer can only experience correlations between these two sets of outcomes and can only predict their

joint probabilities. The causal structure of these two regions, namely wether the correlations have a causal origin or not, is always assumed a priori and cannot be subject to a physical verification. From this fact it follows that if two observers look at one such experiment and for some reason an observer assumes that correlations are due to a causal relationship and the other observer assumes that they are not, they cannot become aware of differences between their respective probabilistic predictions and the joint probabilities originated by the experiment.

Generalizing the result obtained in the above section to experiments involving more than two sets of outcomes presents some subtleties. Consider an experiment involving three sets of random outcomes appearing in three distinct regions of space, say regions A,B,C, such that the outcomes in A cause the outcomes in B and these in turns cause the outcomes in C. Let us suppose that the random outcomes happening in A,B,C are $\{a_i\}$, $\{b_j\}$, $\{c_k\}$ respectively. A physical system $\mathscr{S}$ passing through the three regions constitutes the causal influence propagating from A to B and then from B to C. From an operational point of view $\mathscr{S}$ is the output of region A, the input and the output of region B and the input of region C. An outcome in region B thus represents a possible evolution of $\mathscr{S}$. In quantum theory a system evolution is represented by a CPTP map and is a deterministic notion. The only way to take into account randomness in region B is thus to consider convex combinations of CP maps that decrease the trace of states. An observer assuming an input/output structure of regions A,B,C modified with respect to the one given above, does never arrive to assume A,B,C as three space-like separated regions. Conversely, an experiment where A,B,C are three space-like separated regions and in which the outcomes in the three regions are correlated, is due to a tripartite entangled state. An observer assuming, for this experiment, a different input/output structure, can never arrive to assume that A,B,C are such that outcomes in A cause outcomes in B and that these in turns cause outcomes in C. From these examples we see that when we take into account three regions of space A, B, C, displaying correlated random outcomes, if an observer is able to calculate joint probabilities of the outcomes assuming these three regions as space-like separated, there cannot exist an observer assuming that oucomes on A cause outcomes on B that in turns cause outcomes on C. In order to generalize the result in the previous section to experiments involving more than two sets of random outcomes we thus simply consider that different observers of the same experiment can in principle assume a different input/output structure for the regions involved.

Suppose now to have an experiment in which there are three devices, D_A, D_B, D_C in regions A,B,C respectively displaying random correlated outcomes and that an observer O_2, in order to predict the joint probabilities of the outcomes, assumes that A,B,C are three space-like separated regions. Let the set of outcomes on the three devices be $\{a_i\}_{i\in X} \times \{b_j\}_{j\in Y} \times \{c_k\}_{k\in Z}$ and the associated joint probability distribution be $\{p(a_i, b_j, c_k)\}_{i,j,k\in X\times Y\times Z}$. Let $\mathscr{S}_A, \mathscr{S}_B, \mathscr{S}_C$ be the systems to which the outcomes on D_A, D_B, D_C, refer respectively. O_2 assumes that $\mathscr{S}_A$, $\mathscr{S}_B$, $\mathscr{S}_C$ are respectively three inputs for devices D_A, D_B and D_C. This is represented in figure 5

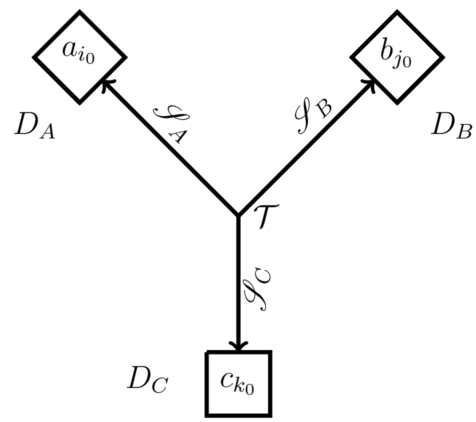

Figure 5

Another observer, O_1, assumes that systems $\mathscr{S}_A$ and $\mathscr{S}_B$ are inputs for D_A and D_B respectively and system $\mathscr{S}_C$ is an output for D_C. This is represented in figure 6.

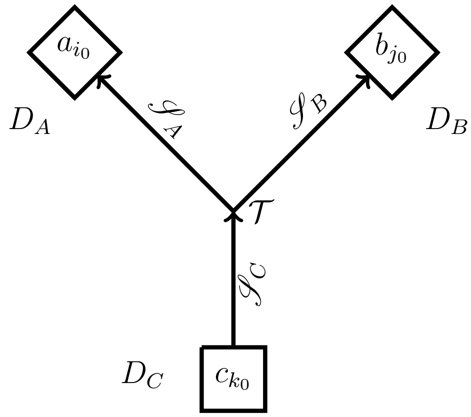

Figure 6

It is easy to see that this situation is not different from the one analyzed in the above sections. O_1 assumes the outcomes on devices D_C as representing preparations belonging to some preparation ensemble represented by a density matrix ρ:

$$\rho = \sum_{k \in Z} \mathrm{Tr}[\mathbf{c_k}\rho] \frac{\sqrt{\rho}\, \mathbf{c_k} \sqrt{\rho}}{\mathrm{Tr}[\mathbf{c_k}\rho]} \tag{16}$$

Moreover he assumes that outcomes on devices D_A and D_B are POVMs $\{\mathbf{a_i}\}_{i\in X}$ $\{\mathbf{b_j}\}_{j\in Y}$. The ensemble ρ undergoes an evolution represented by a CPTP map $\mathscr{T}$ with Kraus decomposition $\sum_m K^m \otimes K^{m\dagger}$ resulting in a density matrix $\mathscr{T}(\rho)$ having the following expression:

$$\mathscr{T}(\rho) = \mathrm{Tr}_C[\mathscr{T}_\rho] \tag{17}$$

where

$$\mathscr{T}_\rho = \sum_{m,ef,cd} K^m_{ef} K^{m*}_{cd} \sqrt{\rho} |c\rangle_{CC}\langle f| \sqrt{\rho} \otimes |e\rangle_{ABAB}\langle d| \tag{18}$$

We see that the only difference between (18) and (8) is that one of the hilbert spaces considered in (18) explicitly refers to the composite system $\mathscr{S}_{AB}$. From the transformation rule stated in the previous section, O_2 assumes that outcomes on devices D_A, D_B and D_C are respectively represened by the POVMs $\{\mathbf{a_i}\}_{i\in X}$, $\{\mathbf{b_j}\}_{j\in Y}$, $\{\mathbf{c_k}^{\mathbf{T}}\}_{k\in Z}$ where ^{T} denotes transposition. The three devices seen by O_2 are indeed correlated by a tripartite entangled state τ_{ABC} that, according to the transformation rule of the previous section, is written as:

$$\tau_{ABC} = \mathscr{T}_\rho^{T_C} \tag{19}$$

O_1 and O_2 experience the same joint probability distribution since:

$$\mathrm{Tr}_{ABC}[\tau_{ABC}\mathbf{a_{i_0}} \otimes \mathbf{b_{j_0}} \otimes \mathbf{c^T_{k_0}}] = \mathrm{Tr}_{AB}[\mathbf{a_{i_0}} \otimes \mathbf{b_{j_0}}\mathrm{Tr}_C[\mathscr{T}_\rho \mathbf{c_{k_0}}]] \tag{20}$$

In the same way they can be treated all the cases in which different observers assume different input/output labels for systems $\mathscr{S}_A$, $\mathscr{S}_B$ and $\mathscr{S}_C$. Based on these arguments it can be seen that analogous results hold for generic experiments in which an arbitrary number of devices display correlated random outcomes.

3.1 Related work

The work presented in the previous section has connections with three other works by Hardy [11], Oreshkov-Costa-Bruckner [12] and Leifer-Spekkens [13]. All these works present formulations of quantum theory in which calculations of joint probabilities for sets of outcomes in distinct regions of space can be performed with a mathematical formalism that is not sensitive of the causal structure imposed to the regions. Note that quantum theory, as is currently regarded, is a formalism that is sensitive to what causal structure is imposed to different correlated regions. For two devices in two regions of space displaying correlated random outcomes such that the outcomes on one device cause those on the other, we have the following mathematical representation: one set of outcomes is represented by a density matrix for a single system that is subject to some evolution represented by a linear map; the other set of oucomes is represented by a set of positive operators that sum to the identity. For two devices displaying correlated random outcomes in two space-like separated regions we have indeed the following mathematical representation: the two sets of outcomes are represented by two sets of positive operators that sum to the identity; a state for the composite system, represented by a density matrix for this system, originates the correlations between the outcomes. On the other hand the analysis done in this section suggests that this may not be the proper way to approach the theory. Indeed, investigating more deeply quantum theory from this point of view we have shown that the two above mathematical representations are more similar than one could expect.

In what follows we review the works in [11, 12, 13]. After this review, we discuss how the results presented in this paper may be connected to these works.

Causaloid

The motivation for the work in [11] is to formulate a framework for quantum gravity. Such a framework should incorporate the radical features of both quantum theory and general relativity. The radical feature of quantum theory is that it is a probabilistic theory. The radical feature of general relativity is that, in this theory, causal structure (represented by the metric tensor) is not fixed rather it is subject to modifications due to changes in the stress-energy tensor representing the energy density of a portion of universe (this is basically the physical content of Einstein's equations). The goal is then to build a framework for probabilistic theories with indefinite (or non fixed) causal structure. Here we will briefly review how quantum theory is formulated in this framework. The new mathematical object introduced in this work is called *causaloid*. A causaloid can be defined in a framework for probabilistic theories generalizing quantum theory. The causaloid is an object that permits to calculate joint probabilities for outcomes happening in different regions of space. In order to specify a causaloid we have to know (i) the physical theory and (ii) the process that originates correlations among the outcomes in the regions of interest. Causaloid specification can be accomplished via a method called *physical compression* of which can be distinguished three different levels. A good example to understand first level physical compression is to think to a quantum state. A quantum state, by definition, is an object containing the information regarding the probabilities of all the possible outcomes appearing in all the measurements performable on the system in a definite region of space. In principle, to specify this object it should be employed an infinite number of real parameters, namely the probabilities for all those outcomes. However it is sufficient to specify a restricted number of real parameters in order to specify a quantum state. For a qubit for example this number is four. This is an example of how first level physical compression is accomplished in quantum theory. Second level of compression is related to the way the theory combines quantities pertaining to two or more distinct regions. Suppose to have two space-like regions and two quantum operators A and B referring to these regions. Then the operator for the composite region is given by $A \otimes B$. In this case we can deduce, from the operator describing the composite region, the operators specifying the component regions. Consider indeed two causally adjacent regions. The way to combine the operators for the composite region starting from the components is $A \circ B$. However if we only know the operator for the composite region we cannot deduce the operators for the component regions.

Hence the number of real parameters that we use to specify $A \circ B$ is less than those we need to specify A and B taken separately. This is compression of the second level. Third level of physical compression is introduced thinking that every physical process can be "sampled" and all the information regarding a process that an observer could ever obtain is contained in quantities pertaining to such sampling regions. Third level of compression thus happens considering the reduction in the number of parameters used to describe a physical process, for example a quantum operation, with respect to the number of parameters needed to specify the process as composed by all the sampling regions.

Causaloid formulation of quantum theory is presented specifying the causaloid for a number of interacting qubits since this kind of process is universal for quantum computation. Such process is sampled in different regions of space $R_1, ...R_T$ by T devices D_1, D_2, ... D_T. The operations performed on the systems by devices D_1, D_2, ... D_T are respectively represented by quantum operations $\$_{\alpha_1}, \$_{\alpha_2}, ..\$_{\alpha_T}$ and are respectively associated to outcomes $\alpha_1, \alpha_2 ... \alpha_T$.

The first level of compression regards each single region R_t. We first consider that such region involves only one qubit. Every quantum operation $\$_{\alpha_t}$ acting on a qubit can be seen as an element of a vector space V_t. It thus can be expanded as a linear combination of other quantum operations forming a spanning set for V_t as:

$$\$_{\alpha_t} = \sum_{l_t} r^{\alpha_t}_{l_t} \$_{l_t} \tag{21}$$

where $\{\$_{l_t}\}_{t=1}^{N}$ is the spanning set for V_t (this is called fiducial set). The matrix for the elementary region R_t given by $\Lambda^{l_t}_{\alpha_t} = r^{\alpha_t}_{l_t}$ is the causaloid for first level of compression in region R_t. We now need to specify the causaloid for regions R_x in which two qubits interact. Consider a region, R_x, in which two qubits, labelled by i and j, interact. Suppose that on these two qubits it acts $\$_{\alpha_x}$ representing some quantum operation for the system composed of the two qubits. $\$_{\alpha_x}$ can be written as a linear combination of a fiducial set of elements in the vector space where live quantum operations acting on the two qubits system. It turns out that the set of product quantum operations $\{\$^i_{l_{xi}} \otimes \$^j_{l_{xj}}\}$ where $\$_{l_{xi}}$ labels a fiducial set of linearly independent quantum operations on qubit i and similarly for j, forms a fiducial set. That is, we

can write

$$\$_{\alpha_x} = \sum_{l_{xi}l_{xj}} r^{\alpha_x}_{l_{xi}l_{xj}} \$^i_{l_{xi}} \otimes \$^j_{l_{xj}} \tag{22}$$

and

$$\Lambda^{l_{xi}l_{xj}}_{\alpha_x} = r^{\alpha_x}_{l_{xi}l_{xj}} \tag{23}$$

represents first level physical compression for a region R_x in which two qubits interact. Now consider second level of physical compression for two causally connected regions R_t, R_{t+1}. Choosing, for each region $\$_1 = I$ (where I is the identity) we can write:

$$\$_{\alpha_{t+1}} \circ \$_{\alpha_t} = \sum_{l_t} r^{\alpha_{t+1}\alpha_t}_{1l_t} I \circ \$_{l_t} \tag{24}$$

since the composition of two quantum operations using $\circ$ is a map on ρ and lives in the same space as a single quantum operation and so we can expand the composition in terms of only one fiducial set of linearly independent quantum operations $\{\$_{l_t}\}^N_{l_t=1}$. Note that to specify the quantum operations acting in the two regions taken separately one needs the coefficients of expansion in V_t on the fiducial set $\{\$_{l_t}\}^N_{l_t=1}$ and the coefficients of expansion in V_{t+1} on the fiducial set $\{\$_{l_{t+1}}\}^N_{l_{t+1}=1}$. This makes a total of N^2 real parameters. In order to specify the composite region indeed one only needs N real parameters as is apparent from (24). Thus we have a reduction of parameters due to second level physical compression. The causaloid for this second level physical compression of those pairs of sequential elementary regions is given by

$$\Lambda^{1k_t}_{l_{t'}l_t} = r^{l_{t'}l_t}_{1k_t} \tag{25}$$

where $t' = t + 1$. The same technique works when we have any number of causally connected regions. Third level physical compression is accomplished by reducing the number of parameters required to specify the causaloid for all the sampling regions of which the process is composed, to the number of parameters required to specify the quantum operation transforming the state of the total system from input to output. The causaloid for this parameters reduction is very complicated but in principle it is possible to write it down. The process we wanted to describe, i.e. pairwise interacting qubits, is thus encodable in a set of matrices relating quantities coming from different regions, and on which the physical theory acts with compression of information. The idea of the causaloid formulation of quantum theory is thus to substitute physical compression in place of pictures such as states of systems

evolving in time that define causal structure in an absolute way. This means formulating quantum theory as a probabilistic theory with indefinite causal structure.

Process matrix

The motivation to introduce the process matrix formulation of quantum theory in [12] is to show that, in this framework, they can be defined processes in which it is not possible to define a causal ordering for the events involved. This is accomplished formulating an inequality for a given game played by two parties that must be satisfyied in every situation in which it is possible to define causal ordering and showing that such inequality can indeed be violated.

They are considered processes in which two or more parties Alice, Bob, Charly ecc. lie in two or more regions of space (or laboratories) A,B,C .. and see random outcomes in their respective regions $\{a\}, \{b\}, \{c\}$.. on which it is defined a joint probability distribution from which the outcomes result correlated. It is assumed that one party, say Alice, can perform all the operations she could perform in a closed laboratory, as described in the standard space-time formulation of quantum theory.These are defined as the set of *quantum instruments* with an input Hilbert space $\mathcal{H}^{A_1}$ (the system coming in) and an output Hilbert space $\mathcal{H}^{A_2}$ (the system going out) (the set of allowed quantum operations can be used as a definition of "closed quantum laboratory" with no reference to a global causal structure). When Alice uses a given instrument, she registers one out of a set of possible outcomes, labeled by $j = 1, \ldots, n$. Each outcome induces a specific transformation from the input to the output, which corresponds to a completely positive (CP) trace-nonincreasing map $\mathcal{M}_j^A : \mathcal{L}(\mathcal{H}^{A_1}) \to \mathcal{L}(\mathcal{H}^{A_2})$, where $\mathcal{L}(\mathcal{H}^X)$, $X = A_1, A_2$, is the vector space of matrices over a Hilbert space $\mathcal{H}^X$ of dimension d_X. The action of each $\mathcal{M}_j^A$ on any matrix $\sigma \in \mathcal{L}(\mathcal{H}^{A_1})$ can be written as $\mathcal{M}_j^A(\sigma) = \sum_{k=1}^m E_{jk}\sigma E_{jk}^\dagger$, $m = d^{A1}d^{A_2}$, where the matrices $E_{jk} : \mathcal{H}^{A_1} \to \mathcal{H}^{A_2}$ satisfy $\sum_{k=1}^m E_{jk}^\dagger E_{jk} \leq I^{A_1}$, $\forall j$. The set of CP maps $\left\{\mathcal{M}_j^A\right\}_{j=1}^n$ corresponding to all the possible outcomes of a quantum instrument has the property that $\sum_{j=1}^n \mathcal{M}_j^A$ is CP and trace-preserving (CPTP), or equivalently $\sum_{j=1}^n \sum_{k=1}^m E_{jk}^\dagger E_{jk} = I^{A_1}$, which reflects the fact that the probability to observe any of the possible outcomes is unity. A CPTP map itself corresponds to an instrument with a single outcome which occurs with certainty. In the case of more than one party, the set of local outcomes cor-

responds to a set of CP maps $\mathcal{M}_i^A, \mathcal{M}_j^B, \cdots$. A complete list of probabilities $P\left(\mathcal{M}_i^A, \mathcal{M}_j^B, \cdots\right)$ for all possible local outcomes will be called *process*. It is explicitly considered only the case of two parties $P(\mathcal{M}_i^A, \mathcal{M}_j^B)$. A *process matrix* for two parties is a mathematical object characterizing the most general probability distribution for two sets of random outcomes $\{i\} \times \{j\}$ corresponding to CP maps $\{\mathcal{M}_i^A\} \times \{\mathcal{M}_j^B\}$. It turns out that the only probabilities $P\left(\mathcal{M}_i^A, \mathcal{M}_j^B\right)$ consistent with the algebraic structure of local quantum operations are bilinear functions of the CP maps $\mathcal{M}_i^A$ and $\mathcal{M}_j^B$. Thus the study of the most general quantum correlations between two distinct devices reduces to the study of bilinear functions of CP maps. Using the Choi-Jamiolkowsky isomorphism one can represent CP maps by means of positive semi-definite matrices. The CJ matrix $M_i^{A_1A_2} \in \mathcal{L}(\mathcal{H}^{A_1} \otimes \mathcal{H}^{A_2})$ corresponding to a linear map $\mathcal{M}_i : \mathcal{L}(\mathcal{H}^{A_1}) \to \mathcal{L}(\mathcal{H}^{A_2})$ is defined as $M_i^{A_1A_2} := [\mathcal{I} \otimes \mathcal{M}_i(|\phi^+\rangle\langle\phi^+|)]^{\mathrm{T}}$, where $|\phi^+\rangle = \sum_{j=1}^{d_{A_1}} |jj\rangle \in \mathcal{H}^{A_1} \otimes \mathcal{H}^{A_1}$ is a (not normalized) maximally entangled state, the set of states $\{|j\rangle\}_{j=1}^{d_{A_1}}$ is an orthonormal basis of $\mathcal{H}^{A_1}$, $\mathcal{I}$ is the identity map, and T denotes matrix transposition. The probability for two measurement outcomes $P(\mathcal{M}_i^A, \mathcal{M}_j^B)$ can thus be expressed as a bilinear function of the corresponding CJ operators as follows:

$$P(\mathcal{M}_i^A, \mathcal{M}_j^B) = \mathrm{tr}[W^{A_1A_2B_1B_2}(M_i^{A_1A_2} \otimes M_j^{B_1B_2})], \tag{26}$$

where $W^{A_1A_2B_1B_2}$ is a matrix in $\mathcal{L}(\mathcal{H}^{A_1} \otimes \mathcal{H}^{A_2} \otimes \mathcal{H}^{B_1} \otimes \mathcal{H}^{B_2})$. The matrix W should be such that probabilities are nonnegative for any pair of CP maps $\mathcal{M}_i^A$, $\mathcal{M}_j^B$. It is required this to be true also for measurements in which the system interacts with any system in the local laboratory, including systems entangled with the other laboratory. This implies that $W^{A_1A_2B_1B_2}$ must be positive semidefinite. Furthermore, the probability for any pair of CPTP maps $\mathcal{M}^A$, $\mathcal{M}^B$ to be realized must be unity (they correspond to instruments with a single outcome). Since a map $\mathcal{M}^A$ is CPTP if and only if its CJ operator satisfies $M^{A_1A_2} \geq 0$ and $\mathrm{tr}_{A_2} M^{A_1A_2} = I^{A_1}$ (similarly for $\mathcal{M}^B$), we can conclude that all bipartite probabilities compatible with local quantum mechanics are generated by matrices W that satisfy

$$W^{A_1A_2B_1B_2} \geq 0 \ \text{[nonnegative probabilities]}, \tag{27}$$

$$\begin{gathered} \mathrm{tr}[W^{A_1A_2B_1B_2}(M^{A_1A_2} \otimes M^{B_1B_2})] = 1, \\ \forall M^{A_1A_2}, M^{B_1B_2} > 0,\ \mathrm{tr}_{A_2} M^{A_1A_2} = I^{A_1}, \mathrm{tr}_{B_2} M^{B_1B_2} = I^{B_1} \\ \text{[sum of probabilities is one]}. \end{gathered} \tag{28}$$

A matrix $W^{A_1A_2B_1B_2}$ that satisfies the above conditions constitutes a process matrix.

We can thus see that also this formalism can be set up to calculate joint probabilities for outcomes pertaining to two regions of space using the same mathematical rules and the same mathematical object independently of the fact that between the regions it is assumed a causal influence or not.

Quantum conditional state

In [13] it is invented the formalism of *quantum conditional states.* Quantum conditional states are used to formulate a theory of Bayesian inference for random variables representing physical observables pertaining to two regions that have a definite causal relationship. The peculiarity of this theory is a tool called *star product.* Star product permits to perform statistical inference for two correlated regions A and B in strict analogy with the ordinary theory of probability in which there is no dependence on the causal relationship between the regions. Quantum conditional states are divided into causal conditional states and acausal conditional states depending on wether the correlations of the outcomes in the two regions are due to a causal relationship or not. A CPTP map, $\mathscr{T}_{AB}$ from region A to region B, is related to an *acausal* conditional state $\rho^s_{A|B}$, by means of the Choi isomorphism [14]:

$$\mathscr{T}_{AB} \leftrightarrow \mathscr{I}_{A'} \otimes \mathscr{T}_{A''B}(|\Phi^+\rangle\langle\Phi^+|) = \rho^s_{\mathrm{A|B}} \tag{29}$$

where $|\Phi^+\rangle = \frac{1}{\sqrt{d_A}} \sum_i |i\rangle_{A'} \otimes |i\rangle_{A''}$ and $\{|i\rangle\}_{i=1}^{d_A}$ is a basis for Hilbert space pertaining to the system in region A and A', A'' two copies of the system in region A. The rule of belief propagation is used to find the joint state ρ^s_{AB} for two systems in space-like separated regions, A and B, starting from the prior pertaining to one of the two regions, ρ_A; this is expressed via the star product:

$$\rho^s_{AB} = \rho_A \star \rho^s_{A|B} = d_A\sqrt{\rho_A} \otimes I_B \; \rho^s_{A|B} \; \sqrt{\rho_A} \otimes I_B \tag{30}$$

The star product used here involves also a normalization factor d_A that cancels with the factor $1/d_A$ arising from the definition of conditional state involving $|\Phi^+\rangle$. The map $\mathscr{T}_{AB}$ is related to a *causal* conditional state $\rho^t_{A|B}$ by means of the Jamiolkowsky isomrphism [15]:

$$\mathscr{T}_{AB} \leftrightarrow [\mathscr{I}_{A'} \otimes \mathscr{T}_{A''B}(|\Phi^+\rangle\langle\Phi^+|)]^{T_{A'}} = \rho^t_{A|B} \tag{31}$$

where ${}^{T_{A'}}$ denotes partial transposition on Hilbert space of system A' pertaining to region A. The rule of belief propagation is used to find the joint state

ρ^t_{AB} for two systems in two causally related regions A and B (or equivalently for one system at two different times) starting from the prior pertaining to region A, ρ_A. This is expressed with the star product as above:

$$\rho^t_{AB} = \rho^T_A \star \rho^t_{A|B} = d_A \sqrt{\rho^T_A} \otimes I_B \; \rho^t_{A|B} \; \sqrt{\rho^T_A} \otimes I_B \tag{32}$$

where ^{T} denotes transposition.

Causaloid, process matrix and quantum conditional state are three different ways of expressing the concept that it is possible, using objects derived from quantum theory, to find mathematical formalisms to calculate joint probabilities of outcomes displayed in distinct regions of space such that the formalism is insensitive of the particular causal structure imposed to the regions. This is achieved through the definition of a single mathematical object (causaloid process matrix and quantum conditional state in the three cases respectively) to perform probability calculations for both types of experiments involving correlations that have a causal origin and involving correlations for space-like separated regions. Our work shows that the standard formalism of quantum theory itself can be seen as such a formalism. For example, the operator defined as $\mathscr{T}_\rho$ in (9), i.e. the evolution by means of map $\mathscr{T}$ of ensemble ρ, has a lot of analogies with a process matrix. Indeed they both represent a way to calculate joint probabilities for outcomes happening in different regions of space that is insensitive to what causal structure is assumed for the regions. This is because the operator τ_{AB} establishing correlations for outcomes in space-like separated regions is a mathematical object of the same nature of $\mathscr{T}_\rho$ (being simply its partial transposition). The main difference between the situation depicted in the previous section and the process matrix formalism is that in the former case, outcomes are represented by POVM elements while in the latter case they are represented by quantum operations. Hence we could regard $\mathscr{T}_\rho$ as a process matrix for POVM elements. There are even more strict similarities with the work in [13]. To see this note that τ_{AB} in (12) is simply ρ^s_{AB} in (30) while $\mathscr{T}_\rho$ in (9) is ρ^t_{AB} in (32). Relationships of the work in [11] with the work presented here (as with the other two works) are less explicit. The work in [11] has the remarkable feature of being formulated in a general probabilistic framework. To achieve such generality it becomes necessarily more abstract and the formulation of quantum theory in this framework suffers of such abstractness. The main idea of the causaloid is however clear and this is that embedding probabilistic

physical processes in space-time (hence giving to probabilistic events a causal structure) is, from an informational/operational point of view, an instance of compression of information. The starting point to reach this conclusion is that causal structure and space-time in physics may not be regarded as something really existing in an objective way. Indeed this is very close to the starting point we adopted in the previous section and to the idea that causal structure of probabilistic outcomes is an observer dependent property.

3.2 Two principles for quantum theory

The work presented here, compared to those reviewed above, has, in our opinion, a deeper foundational value since it poses a new physical principle, the observer dependence of causal structure, as a foundational principle for quantum theory. This is achieved recognizing the equivalence of input/output structure and causal structure and showing that the mathematical formalism of quantum theory is consistent with the assumption that input/output structure is an observer dependent property.

We can thus summarize the work done in this section saying that quantum theory is consistent with the two following principles:

> **Principle of causality** The input/output structure of the devices involved in a quantum experiment defines the causal structure of the outcomes happening on those devices.
>
> **Principle of relativity of causal structure** Two observers looking at a given quantum experiment and assuming a different causal structure for the outcomes involved in the experiment cannot become aware of differences in their respective probabilistic predictions.

In the next section, the principle of relativity of causal structure will be compared with the role causal structure plays in general relativity. In particular it is argued that the situation in general relativity is somewhat opposite to the one outlined above. This is the case since, in general relativity, whether two events in two distinct regions of universe are space-like or not is determined by the metric that, in turn, is related to the stress energy tensor via Einstein's equations. This implies that in general relativity causal structure depends on a (in principle) measurable physical quantity, energy density, and cannot be regarded as an observer dependent property.

4 Causal structure in general relativity

In this section we briefly examine the role causal structure of events has in general relativity. The main equations of general relativity are Einstein's equations relating the metric of a portion of space-time manifold describing a given portion of universe with the mass/energy content of that portion of universe. They are often expressed in the following compact form [16]:

$$G_{\mu\nu} = kT_{\mu\nu} \tag{33}$$

On the r.h.s. k is a constant and $T_{\mu\nu}$ is the stress-energy tensor; on the l.h.s $G_{\mu\nu}$ is the Einstein's tensor and has the following expression:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} \tag{34}$$

where $g_{\mu\nu}$ is the metric, $R_{\mu\nu}$ is the Ricci tensor, R is the Ricci scalar and Λ is the cosmological constant. On a manifold, $(M,\, g)$, a geodesic is a path $x^\mu(\lambda)$ characterized by the following equation [16, 17]:

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda} = 0 \tag{35}$$

In the above equation $\Gamma^\mu_{\rho\sigma}$ are the coefficients of the Levi-Civita connection associated to the metric of the manifold (in general one can use any connection but in general relativity it is used only the Levi-Civita one). This is written as follows:

$$\Gamma^\mu_{\rho\sigma} = \frac{1}{2}g^{\rho\sigma}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}) \tag{36}$$

where ∂_x denotes partial derivative, g_{xy} is the metric and g^{xy} is its inverse. Equation (35) can be interpreted as the vanishing of the covariant derivative of x^μ along the path $x^\mu(\lambda)$. This means that any vector on $x^\mu(\lambda)$ is transported parallel to itself along this path. The tangent vector to a point of the geodesic describes an interval between two points in the tangent space. If the manifold is a solution of Einstein's Equations, such interval can be time-like, null or space-like depending on the norm of the vector. Since a geodesic describes a path along which a tangent vector of the manifold is parallel transported, we have that if the tangent vector on a given point of the geodesic is time-like, null or space-like, the tangent vector on any other

point of the geodesic will preserve this property. From this, one interprets geodesics where the tangent vector is time-like or null as paths followed respectively by freely falling material particles or photons. On the other hand if the tangent vector is space-like, then there is no physical system that can follow the path corresponding to the geodesics. From this fact we can state that, in general relativity, given two points in space-time x_a, x_b, pertaining to two different regions of universe R_A, R_B respectively, it can exist a causal relationship between them (namely it is possible for a material or light particle to start at x_a and cause an effect at x_b) if the two points lie on a time-like or null geodesic. On the other hand it cannot exist a causal relationship between the two points if they lie on a space-like geodesic. From (35) and (36) we see that, in last analysis, the metric tensor is the object characterizing geodesics. This together with Einstein's equations imply that the stress-energy tensor representing the energy density in a given portion of universe establishes whether between two space-time points it can exist a causal relationship.

According to general relativity we thus have that the existence (or non existence) of a causal relationship between two events depends on the energy density of the portion of universe in which the events happen and thus on an objective physical quantity. This means that causal structure in general relativity should (in principle) be inferred in an objective way by whatever observer by means of energy density measurements. We used the conditional because it is well known that, on large cosmological scales, to explain at best observational data it must be introduced dark energy and this poses various problems from the theoretical point of view. In the following section we will briefly review these problematic issues. After that we will discuss the possible relationship that could exist between these problems in general relativity and the fact that causal structure in quantum theory may be regarded as an observer dependent property.

5 Dark energy

In this section we first give a brief review on dark energy. This material is mostly taken from a review on the subject done by Carroll [18]. We then discuss the conclusions reached in this review in relationship with the observer dependence of causal structure for outcomes happening in quantum experiments.

The standard assumption in cosmology is that universe is homogenous and isotropic. Since in general relativity, universe is described by a manifold M, these two assumtpions translate into formal statements regarding the geometry of M. Homogeneity means that given two points p, q in M there exists an isometry that takes p into q. Isotropy means that given a point p in M, for any two vectors v and w in T_pM, there exists an isometry such that the pushforward of w under the isometry is parallel to v. Since the universe is not static, we infer that it is homogeneous and isostropic in space but not in time. This and the above assumptions imply that the universe can be foliated in space-like slices such that each slice is homogeneous and isotropic. Based only on these considerations it can be shown [16, 17] that the metric of the universe must have the following form:

$$ds^2 = -dt^2 + a^2(t) d\sigma_3^2(k) \tag{37}$$

where $a(t)$ is the scale factor and $d\sigma_3^2(k)$ is a metric for three space which depends on the curvature parameter k. The metric in (37) is called the Friedmann Robertson Lemaitre Walker (FRLW) metric. Note that Einstein's equations are not taken into account to derive (37) since its derivation is based on purely geometrical arguments. Einstein's equations are used to find the functional form for $a(t)$. In order to do so it must be made the assumption that matter and energy on large cosmological scale can be modelled as a perfect fluid and it is choosen an equation of state relating pressure p to matter and energy density ρ of the type $p = w\rho$ with w constant. Putting the metric in (37) into Einstein's equations and using the above assumption leads to write Friedman equations [16, 17], i.e. a set of differential equations establishing the evolution of scale factor in relationship with curvature, pressure and energy density:

$$\frac{\ddot{a}}{a} = \frac{4\pi G}{3}(\rho + 3p) \tag{38}$$

$$(\frac{\dot{a}}{a})^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} \tag{39}$$

The quantity on the l.h.s of (39) is the square of the *Hubble parameter* $H = \frac{\dot{a}}{a}$ and can be used to define the value of the *critical density*:

$$\rho_c = \frac{3H^2}{8\pi G} \tag{40}$$

The critical density is the value of energy density solving Friedman's equations for zero spatial curvature, i.e. for a flat universe. Exploiting ρ_c one can define the *density parameter* $\Omega = \frac{\rho}{\rho_c}$ by means of which (39) can be written as:

$$\Omega - 1 = \frac{k}{H^2 a^2} \tag{41}$$

This shows that whether $k = +1, 0, -1$ depends on the magnitude of the actual (i.e. observed) energy density ρ with respect to critical density ρ_c. If $\Omega < 1$ then $k < 0$ and the universe is described by a three dimensional manifold with constant negative curvature. On the contrary, if $\Omega > 1$ then $k > 0$ and the universe is decribed by a three dimensional manifold with constant positive curvature (the analog in three dimension of a sphere). Finally $\Omega = 1$ implies $k = 0$ and describes a flat universe the associated manifold being simply a three dimensional euclidean space.

There are three forms of energy density usually considered. The first is called *dust* ρ_d and is composed of non relativistic matter whose pressure is negligible with respect to its energy density. The second is called *radiation* ρ_r and is composed of photons and other relativistic particles moving approximately at the speed of light. The third is *dark energy* ρ_Λ coming from the introduction of the cosmological constant in Einstein's equations. There are strong evidences [18] that the amount of total energy density ρ due to dust is negligible with respect to the amount due to matter ($\rho_m/\rho_d = 10^6$). We thus say that we live in a matter dominated universe and the relevant contributions to total energy density come from ρ_d and ρ_Λ.

Observations of the dynamics of galaxies and clusters have shown that a reasonable value for the density parameter referring to ρ_d, is $\Omega_d = 0.3 \pm 0.1$ [19]. On the other hand observations of the anisotropies of the cosmic microwave background are consistent with a nearly spatially flat universe [19]. Thus we infer $\Omega \approx 1$. This implies that the amount of ρ_Λ to the total energy density is such that $\Omega_\Lambda \approx 0.7$. Measurements of the distance vs. redshift relation for Type Ia supernovae [20, 21] have provided evidences that the universe is accelerating i.e. that $\ddot{a} > 0$. Since conventional matter could not make the universe expansion accelarate it is inferred that the component of the energy density that is responsible for such acceleration is ρ_Λ. The most natural candidate component of energy density for ρ_Λ is the vacuum energy ρ_v. This is corroborated by the following argument. Let us write (39) as:

$$\dot{a}^2 = \frac{8\pi G}{3} a^2 \rho - k. \tag{42}$$

If the universe is expanding, then ρ_d must necessarily decrease as the particle number density is diluted by expansion, so $\rho_d \propto a^{-3}$. Hence the right-hand side of (42) will be decreasing in an expanding universe (since $a^2\rho$ is decreasing, while k is a constant), hence the derivative of $\dot{a}$ should be negative if one only takes into account the contribution of ρ_d. The supernova data therefore imply that, to make the universe accelerate, there must be a source of energy density that varies more slowly than $a^2\rho$ i.e. more slowly than a^{-2}. Since the distinguishing feature of vacuum energy is that it is a minimum amount of energy density in any region, strictly constant throughout spacetime, the slow variation of ρ_Λ corroborates the statement that vacuum energy be the source of energy density making the expansion of universe accelerate. To match the data, it is required a vacuum energy:

$$\rho_v \approx (10^{-3}\mathrm{eV})^4 = 10^{-8}\mathrm{ergs/cm}^3 \tag{43}$$

It is not possible to reliably calculate the expected vacuum energy in the universe, or even in some specific field theory such as the Standard Model of particle physics; at best they can be evaluated order-of-magnitude estimates for the contributions from different sectors. These estimates lead to the following value:

$$\rho_v^{(\mathrm{theory})} \sim (10^{27}\ \mathrm{eV})^4 = 10^{112}\ \mathrm{ergs/cm}^3\ . \tag{44}$$

This value is 120 orders of magnitude (30 if we change units of measurement) greater than the value in (43). Such a huge discrepancy with observational data implies that the source of energy density responsible for the expansion of universe, ρ_Λ, should be something different from the vacuum energy. This is known as the *cosmological constant problem*.

As already told the actual model for the universe has $\Omega_\Lambda = 0.7$ and $\Omega_d = 0.3$ but the relative balance of dark energy and matter changes rapidly as the universe expands:

$$\frac{\Omega_\Lambda}{\Omega_d} = \frac{\rho_\Lambda}{\rho_d} \propto a^3 \tag{45}$$

This is due to the facts pointed out above, namely, that ρ_Λ should be almost constant while $\rho_d \propto a^{-3}$. As a consequence, at early times of the universe's expansion, dark energy was negligible in comparison to matter and radiation, while at late times matter and radiation are negligible. There is only a brief epoch of the universe's history during which it would be possible to witness the transition from domination by one type of component to another. On

the other hand, from the fact that $\Omega_\Lambda = 0.7$ and $\Omega_d = 0.3$ we conclude that we actually live in such a transitional period. It seems remarkable that we live during the short transitional period between those two eras. The approximate coincidence between matter and dark energies in the current universe is called the *coincidence problem*.

Inferring the existence of a source of energy different from ordinary matter or radiation to explain observational data in cosmology is not, on its own, a conceptual problem. Problems arise because it is not possible to explain the origin of this source of energy in a scenario that is logically consistent with the current physical knowledge. Thus, the problematic issues of inferring the existence of dark energy lie in the fact that this inference leads to logical inconsistencies such as the cosmological constant problem and the coincidence problem.

In general relativity, the metric encodes the information regarding the causal structure of space-time events. Since the metric tensor is determined by energy density via Einstein's equations and this is a physical quantity that has an objective value, we have that causal structure in general relativity is an objective property. On the other hand, in section 3 we showed that in quantum theory, causal structure of events happening in experiments is an observer dependent property. This is the case since there is no way to physically distinguish whether two sets of probabilistic events happen in two causally related regions or in two space-like separated regions. Hence the situation in quantum theory is somewhat opposite to that of general relativity. Current observations of the universe at large cosmological scale (such as redshifts in spectra emitted from far away galaxies and clusters) are interpreted according to general relativity and this leads to several conceptual problems (two of which are reported above). It is remarkable that these difficulties in general relativity are encountered in quantifying sources of energy density that should justify the presumed flat metric ($k = 0$) of the universe. This is as saying that these difficulties arise when we try to relate a physical objective quantity, energy density, to the metric tensor, encoding the information on the causal structure of space-time events. Hence when we try to determine causal structure of space-time events as if it was objectively established. These facts suggest that elevating the principle of relativity of causal structure to a universal principle could be connected to the problem of dark energy. Indeed, if dark energy was not necessary to explain cosmological observations and we could estimate the sources of energy responsible for the

inferred dynamics of the universe, then, in principle, two observers could not assume different perspectives regarding the existence of a causal connection between two regions of universe since this would be absolutely defined by energy density measurements. Elevating the principle of relativity of causal structure to a universal principle thus poses dark energy not as a conceptual problem but as an essential ingredient of our current understanding of the universe. However, in doing this, it should be faced the deepest problem of searching a theory of gravity completely different from general relativity that possibly reduces to general relativity in appropriate limits.

6 Conclusions

Quantum theory is an extraordinarily successful theory and still lacks a clear physical explanation. Moreover, the absence of any experiments linking quantum theory with the geometry of space-time leaves physicists with the consciousness that something is missing in our current understanding of nature at a fundamental level. This has renewed efforts in finding foundational principles for quantum theory in order to find a more general theory.

In this paper it is analyzed the interplay between causal structure of space-time events and the probabilistic nature of quantum theory. This analysis leads us to state two principles that can be put as foundations of quantum theory:

> **Principle of causality** The input/output structure of the devices involved in a quantum experiment defines the causal structure of the outcomes happening on those devices.
>
> **Principle of relativity of causal structure** Two observers looking at a given quantum experiment and assuming a different causal structure for the outcomes involved in the experiment cannot become aware of differences in their respective probabilistic predictions.

Since the only thing that can be predicted and physically verified in quantum theory are probabilities, the last principle suggests that causal structure of outcomes happening in quantum experiments is an observer dependent property. This principle could be a guiding principle to construct a theory of quantum gravity for the following reason. Quantum theory and general

relativity are both successful and problematic in different and somewhat opposite aspects. On one hand quantum theory is extremely successful in making predictions. Until now, no experimental situation has been found in which the predictions of quantum theory are not satisfied. However there are still difficulties, after almost 90 years from its birth, to understand its physical meaning. On the other hand general relativity is not completely satisfactory in making predictions at large cosmological scales. This is related to the need to introduce dark energy to explain observational data. General relativity is, by the way, founded on two extremely clear and intuitive physical principles, namely, the Einstein's principles of relativity and equivalence. It is then likely that a theory more fundamental than the ones we have at the moment will come from a physical principle that can be put as foundation of quantum theory on one hand and that can motivate the need to introduce dark energy to explain observational data at large cosmological scales on the other hand. The principle of relativity of causal structure is indeed such a principle as we discussed in the previous section. If dark energy was not necessary to explain cosmological observations, and we could estimate the sources of energy responsible for the inferred dynamics of the universe, then, in principle, two observers could not assume different perspectives regarding the existence of a causal connection between two regions of universe since this would be absolutely defined by energy density measurements. Elaborating a theory of quantum gravity starting from the conclusions of this work is an extremely hard task and its success is far from being certain. The main motivation to try to formulate a new theory according to the above principle is that, as far as we know, the most plausible proposal for a source of dark energy is the assumption of a "cosmic aether" permeating all space whose origin is unknown [22]. Clearly this cannot be satisfactory since we are forcing new physical degrees of freedom, motivated only by the fact that the current model of universe and the theory underlying it do not properly explain observations.

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